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Virialization of matter overdensities within dark energy subsystems: special cases

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Abstract

The virialization of matter overdensities within dark energy subsystems is considered under a number of restrictive assumptions, namely (i) spherical-symmetric density profiles, (ii) time-independent quintessence equation of state parameter, w, and (iii) nothing but gravitational interaction between dark energy scalar field and matter. In addition, the quintessence subsystem is conceived as made of "particles" whose mutual interaction has intensity equal to $G(1+3w)$ and scales as the inverse square of their distance. The related expression of the self and tidal potential energy and formulation of the virial theorem for subsystems, are found to be consistent with their matter counterparts, passing from $-1 \le w < -1/3$ to $w = 0$. In the special case of fully clustered quintessence, energy conservation is assumed with regard to either the whole system (global energy conservation), or to the matter subsystem within the tidal potential induced by the quintessence subsystem (partial energy conservation). Further investigation is devoted to a few special cases, namely a limiting situation, $w = -1/3$, and three lower values, $w = -1/2, -2/3, -1$, where the last one mimics

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the effect of a cosmological constant. The special case of fully clustered (i.e. collapsing together with the matter) quintessence is studied in detail, using a similar procedure as in Maor & Lahav (2005). The general case of partially clustered quintessence is considered in terms of a degree of quintessence de-clustering, ζ, $0 \leq \zeta \leq 1$, ranging from fully clustered ($\zeta = 0$) to completely de-clustered ($\zeta = 1$) quintessence, respectively. The special case of unclustered (i.e. remaining homogeneous) quintessence is also discussed. The trend exhibited by the fractional (virialization to turnaround) radius, η, as a function of the (i) fractional (quintessence to matter) "mass" ratio at turnaround, m, (ii) degree of quintessence de-clustering, ζ, and (iii) quintessence equation of state parameter, w, is found to be different from its counterparts reported in earlier attempts. In particular, no clear dichotomy with respect to the limiting situation of vanishing quintessence, $\eta = 1/2$, is shown when global or partial energy conservation hold in the special case of fully clustered quintessence, with $\eta > 1/2$ preferred. The reasons of the above mentioned discrepancy are recognized as owing to (i) different formulations of the virial theorem for subsystems, and (ii) different descriptions of de-clustered quintessence, with respect to the reference case of fully clustered quintessence.
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1 Introduction

Recent observations from anisotropies in the cosmic microwave background, large-scale structure surveys, Hubble parameter determinations, and Type Ia supernova results, allow narrow ranges for the values of cosmological quantities (e.g., Sievers et al. 2003; Rubigño-Martin et al. 2003; Spergel et al. 2003). The related "corcondance" cosmological model is consistent with a bottom-up picture (hierarchical clustering) of dark matter haloes, where smaller systems formed first from initial density perturbations and then merged with each other to become larger systems, or were tidally disrupted and accreted from successively formed, more massive neighbours. A main feature of the concordance model is that the dominant (about 70%) contribution to the present-day energy budget is a component with equation of state, $p = wc^2\rho$, called dark energy, where p is the pressure, ρ the energy density, c the speed of the light, and w a dimensionless parameter which is, in general, time-dependent.

An obvious candidate, a scalar field, has to be light enough to vary slowly during a Hubble time, in such a way its potential energy can drive an accelerated expansion, just like during inflation. The varying field equation of state can then be tuned to lie in the observed range, and the related scalar field is sometimes called "quintessence", which may be conceived as a fifth component of the cosmic fluid after ordinary matter (baryons, leptons, and radiation) and nonbaryonic dark matter. The scalar field density fraction, Ω_q, can be made to decrease rapidly in the past, so as to pass easily the lensing constraints and to avoid discrepancies in the primordial nucleosynthesis abundances. For further details, see e.g., Amendola (2000).

The parameter of the equation of state, w, depends on how the scalar field is slowly rolling down its potential. In the limit of a completely flat potential, $w = -1$, the quintessence behaves as a cosmological constant, $\Lambda > 0$ (Wetterich 1988; Peebles & Ratra 1988; Ratra & Peebles 1988). A subclass of quintessence models with constant w was proposed by Caldwell et al. (1998a,b) within the range, $-1 \leq w < -1/3$. In fact, $w > -1/3$ corresponds to decelerate expansion while $w < -1$ implies a phisical interpretation which is not still clear up today (e.g., Maor & Lahav 2005, hereafter quoted as ML05). Data on large-scale structures suggest a preferential range, $-1 \leq w \leq -0.6$ (e.g., Weinberg & Kamionkowski 2003). Many other forms have been proposed for the shape of the potential of the scalar field, leading to an equation of state parameter that is dependent on the scale factor (see Peebles & Ratra 2003, for a review). For further details see e.g., Percival (2005).

Dark energy affects not only the expansion rate of the background and the distance-redshift relation, but also the growth of structure. The formation rate of haloes, their evolution and their final characteristics are modified. Dark energy is therefore expected to have an impact on observables such as cluster number counts and lensing statistics due to intervening concentrations of mass on the line of sight of background sources. For further details see e.g., Horellou & Berge (2005). The assumption of an Einstein-de Sitter cosmology, $\Omega_m + \Omega_q = 1$, according to the concordance model, allows a much simpler analysis, but additional assumptions are needed.

First, overdensities are conceived as spherical-symmetric, as initially done by Gunn & Gott (1972). The related top hat spherical collapse formalism has been proven to be a powerful tool for understanding the formation and the evolution of bound systems in the universe. For reasons of simplicity, the special case of homogeneous overdensities is currently considered (e.g., Lahav et al. 1991; Wang & Steinhardt 1998; Iliev & Shapiro 2001; Horellou & Berge 2005; Percival 2005; ML05).

Second, the quintessence equation of state is restricted to constant values of the parameter, w. A time-dependent parameter, w, can result from a changing ratio of quintessence kinetic to potential energy which, in turn, is owing to the evolution of the scalar field potential (e.g., Caldwell et al. 1998b). Though time-varying equations of state have been used in the literature (e.g., Wetterich 1995; Amendola 2000; Battye & Weller 2003; Mota & van de Bruck 2004; Percival 2005), the special case of constant w makes considerable simplification (e.g., Caldwell et al. 1998b; Wang & Steinhardt 1998; Weinberg & Kamionkowski 2003; Horellou & Berge 2005; ML05).

Third, any couplings of quintessence to other fields are supposed to be negligibly small, so that the scalar field interacts with other matter only gravitationally. Different interactions can be taken into consideration (e.g., Wetterich 1995; Amendola 2000) at the price of less simple analysis involving a large number of parameters.

In this view, the quintessence is conceived as lying between two limiting cases, namely (i) full clustering i.e. the scalar field responds to the infall in the same way as matter, and (ii) unclustering i.e. its sole effect is a tidal potential acting on the matter overdensity. The general case of partial clustering can be taken into consideration, at the expense of a much more complicated continuity equation for the quintessence overdensity (e.g., Mota & van de Bruck 2004; ML05). Even though it has been shown that quintessence cannot be perfectly smooth (e.g., Caldwell et al. 1998a,b), clustering is usually assumed to be negligible on scales less than about 100 Mpc (e.g., Wang & Steinhardt 1998; Weinberg & Kamionkowski 2003; Battye & Weller 2003; Horellou & Berge 2005). It is therefore common practice to keep the quintessence homogeneous during the evolution of the system. The effects of relaxing this assumption were explored in recent attempts (e.g., Mota & van de Bruck 2004; Percival 2005). The real situation may safely be expected to lie between two limiting cases, where the quintessence is clustering together with the matter and remains homogeneous, respectively.

In spite of the simplifying assumptions mentioned above, still different authors use different expressions for (i) the gravitational potential induced by the quintessence, (ii) the self and the tidal potential energy, and (iii) the formulation of the virial theorem, which yields different results. On the other hand, considerable effort has been devoted to the fact, that the quintessence is indistinguishable from a cosmological constant in the special case, $w = -1$ (e.g., Wang & Steinhardt 1998; Mota & van de Bruck 2004; Horellou & Berge 2005; Percival 2005; but see also ML05; Wang 2006), but little consideration has been taken on the fact, that the quintessence is indistinguishable from matter in the special case, $w = 0$ (e.g., Caldwell et al. 1988b; Iliev & Shapiro 2001). Accordingly, the expression of the gravitational potential induced by the quintessence subsystem within a density perturbation, is expected to reduce to its matter counterpart as $w \to 0$. In fact, quintessence and matter gravitational potential are sometimes unified in a single expression, where $w < -1/3$ corresponds to the former case and $w = 0$ to the latter (e.g., ML05).

The current paper is restricted to the virialization of matter overdensities within quintessence subsystems, under the simplifying assumptions mentioned above and, in addition, the boundary condition that the quintessence potential reduces to the gravitational potential, when the pressure term is suppressed. The gravitational potential and potential energy terms for matter and quintessence subsystem, are expressed in Sect. 2. The virial theorem for 2-component systems is specified in Sect. 3. The special case of fully clustered quintessence, and its extension to the general case of partially clustered quintessence, are investigated in Sects. 4 and 5, respectively. The related discussion is performed in Sect. 6, and some concluding remarks are reported in Sect. 7. Further investigation on a few special arguments is developed in the Appendix.

2 Matter and quintessence potential-energy terms

Let us take into consideration homogeneous and spherical-symmetric overdensities made of both matter and quintessence. Let the quintessence equation of state:

$$p_q = wc^2\rho_q \quad ; \tag{1}$$

be restricted to constant values of the parameter, w, being p the pressure, ρ the density, c the light velocity in vacuum, and the index, q, denoting quintessence. Let any couplings of quintessence to other fields be negligibly small, so that the scalar field interacts with other matter only gravitationally.

The overdensity may be conceived as a two-component system made of matter and quintessence, respectively. The related gravitational potential is (e.g., Mota & van de Bruck 2004; ML05)[1]:

$$\mathcal{V}_u(r) = 2\pi G(1+3w_u)\rho_u \left(R^2 - \frac{r^2}{3}\right) \quad ; \qquad u = m, q \quad ; \tag{2}$$

where $w_m = 0$, $w_q = w$, R is the radius of the overdense sphere, and the index, m, denotes matter.

With regard to matter component, Eq. (2) reduces to the well known expression of the gravitational potential induced by a homogeneous sphere on interior points, with the boundary condition that the maximum value is attained at the centre (e.g., MacMillan 1930, Chap. II, § 29). In fact, given an interior point placed at a distance, r, from the centre, the potential induced from the sphere of radius, r, is $\mathcal{V}^{(\mathrm{int})}(r) = (4\pi/3)G\rho r^2$, according to MacLaurin's theorem, and the potential induced from the corona of radii, r and R, is $\mathcal{V}^{(\mathrm{ext})}(r) = 2\pi G\rho(R^2 - r^2)$, according to Newton's theorem, and the potential induced by the sphere of radius, R, at the point under consideration, is $\mathcal{V}(r) = \mathcal{V}^{(\mathrm{int})}(r) + \mathcal{V}^{(\mathrm{ext})}(r)$, according to Eq. (2) particularized to the matter subsystem. For further details see e.g., Caimmi (2003). An alternative expression is related to the boundary condition of a null potential at the centre (e.g., Iliev & Shapiro 2001; Horellou & Berge 2005).

In general, the gravitational potential may be conceived as induced from a matter distribution where any two particles, idealized as mass points, interact with strenght, G, according to Newton's law, $F_{m_i m_j} = G m_i m_j / r_{ij}^2$, and the dependence of the resulting force on the distance, $\partial\mathcal{V}/\partial r \propto r$, deduced from Eq. (2), is owing to the selected density profile, $\rho = \mathrm{const}$.

In this view, the gravitational potential induced by the quintessence may also be conceived as arising from a distribution where any two "particles", idealized as "mass points", interact with strenght, $(1 + 3w)G$, according to a Newton-like law, $F_{m_{qi} m_{qj}} = (1 + 3w) G m_{qi} m_{qj} / r_{ij}^2$, where m_q is the "quintessence mass". Then the results related to two-component matter distributions (e.g., Limber 1959; Brosche et al. 1983; Caimmi et al. 1984; Caimmi & Secco 1992) may be generalized to the case, where a subsystem is made of quintessence.

[1]A different sign convention is adopted here.

The self potential energy reads (e.g., Chandrasekhar 1969, Chap. 2, § 10):

$$\Omega_u = -\frac{1}{2}\int_{S_u} \rho_u(x_1,x_2,x_3)\mathcal{V}_u(x_1,x_2,x_3)\,\mathrm{d}^3S \quad ; \qquad u = m, q \quad ; \tag{3}$$

where S_u is the volume of u subsystem.

The interaction potential energy, the tidal potential energy, and the residual potential energy read (e.g., Caimmi & Secco 1992):

$$W_{uv} = -\frac{1}{2}\int_{S_u} \rho_u(x_1,x_2,x_3)\mathcal{V}_v(x_1,x_2,x_3)\,\mathrm{d}^3S \quad ; \tag{4}$$

$$V_{uv} = \int_{S_u} \rho_u(x_1,x_2,x_3)\sum_{r=1}^{3} x_r \frac{\partial \mathcal{V}_v}{\partial x_r}\,\mathrm{d}^3S \quad ; \tag{5}$$

$$Q_{uv} = V_{uv} - W_{uv} \quad ; \qquad u = m, q \quad ; \qquad v = q, m \quad ; \tag{6}$$

respectively.

The combination of Eqs. (2) and (4) yields:

$$\frac{W_{uv}}{1+3w_v} = \frac{W_{vu}}{1+3w_u} \quad ; \qquad u = m, q \quad ; \qquad v = q, m \quad ; \tag{7}$$

in the special case of subsystems obeying the same equation of state, $w_u = w_v$, the interaction potential energy is symmetric with respect to the exchange of indices (e.g., MacMillan 1930, Chap. III, §76; Caimmi & Secco 1992; Caimmi 2003).

In the special case of concentric, spherical-symmetric, homogeneous subsystems, the gravitational potential is expressed by Eq. (2), and Eqs. (3), (4), (5), and (7) reduce to (e.g., Caimmi 2003):

$$\Omega_u = -\frac{16}{15}\pi^2(1+3w_u)G\rho_u^2R_u^5 = -\frac{3}{5}(1+3w_u)\frac{GM_u^2}{R_u} \quad ; \qquad u = m, q \quad ; \tag{8}$$

$$W_{uv} = -\frac{3}{5}(1+3w_v)\frac{GM_i^2}{R_i}\frac{m}{y^3}\left(\frac{5}{4}y^2 - \frac{1}{4}\right) \quad ; \quad u = m, q \quad ; \quad v = q, m \quad ; \tag{9}$$

$$V_{ij} = -\frac{3}{5}(1+3w_j)\frac{GM_i^2}{R_i}\frac{m}{y^3} \quad ; \tag{10}$$

where the "mass" of the quintessence subsystem, M_q, and the dimensionless ratios, m and y, are defined as:

$$M_q = \frac{4\pi}{3}R_q^3\rho_q \quad ; \tag{11}$$

$$m = \frac{M_j}{M_i} \quad ; \qquad y = \frac{R_j}{R_i} \quad ; \qquad R_i \le R_j \quad ; \tag{12}$$

and the indices, i and j, denote the inner and the outer component, respectively.

Using the virial theorem for both subsystems and the whole system, together with Eqs. (7), (9) and (10), the tidal potential energy, V_{ji}, takes the explicit expression:

$$V_{ji} = -\frac{3}{5}\frac{GM_i^2}{R_i}\frac{m}{y^3}\left(\frac{5}{2}y^2 - \frac{3}{2}\right) \times \left[1 + \frac{3}{2}w_j\frac{5y^2-5}{5y^2-3} + \frac{3}{2}w_i\frac{5y^2-1}{5y^2-3}\right] \quad ; \tag{13}$$

and a formal demonstration is provided in the next section. The combination of Eqs. (10) and (13) yields:

$$V_{ji} = V_{ij}\left[\left(\frac{5}{4}y^2 - \frac{5}{4}\right) + \left(\frac{5}{4}y^2 - \frac{1}{4}\right)\frac{1+3w_i}{1+3w_j}\right] \quad ; \tag{14}$$

which makes a relation between tidal potential energies.

Under the further restriction that the two subsystems are bounded by a single sphere, $R_i = R_j = R$ or $y = 1$, Eqs. (9), (10) and (14) reduce to:

$$\frac{W_{mq}}{1+3w_q} = \frac{W_{qm}}{1+3w_m} = -\frac{3}{5}\frac{GM_mM_q}{R} \quad ; \tag{15}$$

$$\frac{V_{mq}}{1+3w_q} = \frac{V_{qm}}{1+3w_m} = -\frac{3}{5}\frac{GM_mM_q}{R} \quad ; \tag{16}$$

and the interaction potential energy coincides with the tidal potential energy, which is the limiting case currently used in the literature (e.g., Lahav et al. 1991; Wang & Steinhardt 1998; Eliev & Shapiro 2001; Horellou & Berge 2005; Percival 2005; ML05).

Also it is worth of note (Percival 2005) that there is some confusion in the literature about the exact form of $V_{mq} = W_{mq}$, and the $(1 + 3w_q)$ term has sometimes been neglected in the past, although it is included im more recent work (Battye & Weller 2004; Horellou & Berge 2005; Percival 2005; ML05). Owing to a different choice of the potential, the current expression of V_{mq} coincides with its counterpart calculated in ML05, while the corresponding result appearing elsewhere (e.g., Battye & Weller 2004; Horellou & Berge 2005; Percival 2005) is different by a factor, $-1/2$.

3 The virial theorem for matter and quintessence two-component systems

It is worth recalling that the potential induced by the quintessence has been interpreted (Sect. 2) in terms of an interaction of strenght, $(1 + w_q)G$, which depends on the inverse square distance. Accordingly, the virial theorem for the whole system reads (e.g., Landau & Lifchitz 1966, Chap. II, § 10):

$$2T + \Omega = 0 \quad ; \tag{17}$$

where Ω is the potential energy (e.g., MacMillan 1930, Chap. III, § 76; Caimmi & Secco 1992):

$$\Omega = \Omega_m + W_{mq} + W_{qm} + \Omega_q \quad ; \tag{18}$$

and T is the kinetic energy.

On the other hand, the virial theorem for subsystems reads (e.g., Limber 1959; Brosche et al. 1983; Caimmi et al. 1984; Caimmi & Secco 1992; Caimmi 2003):

$$2T_u + \Omega_u + V_{uv} = 0 \quad ; \qquad u = m, q \quad ; \qquad v = q, m \quad ; \tag{19}$$

where T_u is the kinetic energy of u-th subsystem.

At this stage, let us assume that the quintessence subsystem can retain some form of kinetic energy, in such a way it is allowed to virialize by itself, within the tidal potential induced by the matter subsystem. In this view, the substitution of Eq. (18) into (17) yields:

$$2T_m + 2T_q + \Omega_m + \Omega_q + W_{mq} + W_{qm} = 0 \quad ; \tag{20}$$

and the summation of the left-side member of Eq. (19) with its counterpart where the indices, u and v, are interchanged, produces:

$$2T_m + 2T_q + \Omega_m + \Omega_q + V_{mq} + V_{qm} = 0 \quad ; \tag{21}$$

finally, the combination of Eqs. (20) and (21) yields:

$$V_{mq} + V_{qm} = W_{mq} + W_{qm} \quad ; \tag{22}$$

or, without loss of generality:

$$V_{uv} = W_{uv} + Q_{uv} \quad ; \qquad u = m, q \quad ; \qquad v = q, m \quad ; \tag{23}$$

$$Q_{mq} = -Q_{qm} \quad ; \tag{24}$$

where $Q_{uv} = V_{uv} - W_{uv}$ is the residual potential energy (e.g., Caimmi & Secco 1992).

In the special case of concentric, spherical-symmetric, homogeneous subsystems, the combination of Eqs. (9), (10), and (23) yields:

$$Q_{ij} = -\frac{3}{5}(1+3w_j)\frac{GM_i^2}{R_i}\frac{m}{y^3}\left(\frac{5}{4}-\frac{5}{4}y^2\right) \quad ; \tag{25}$$

and the combination of Eqs. (23) and (24) produces:

$$V_{ji} = W_{ji} + Q_{ji} = W_{ji} - Q_{ij} \quad ; \tag{26}$$

finally, using Eqs. (9) and (25) makes Eq. (26) coincide with (13). The further restriction that the two subsystems are bounded by a single sphere, $R_i = R_j = R$ or $y = 1$, implies $Q_{uv} = 0$ and then $V_{uv} = W_{uv}$.

The virial theorem, as expressed by Eqs. (17) and (19), is different from $2T + \Omega_m + 2V_{mq} = 0$ currently used in the literature (e.g., Horellou & Berge 2005; Percival 2005), due to the subtraction of Ω_q or the addition of V_{mq}, respectively. On the other hand, the formulation $2T - R\partial(\Omega_m + V_{mq} + V_{qm} + \Omega_q)/\partial R = 0$ used in ML05 is valid for the whole system, but cannot be used for subsystems. For further details, see Appendix A.

In what follows, it shall be intended that the two fluids are made of matter ($w_m = 0$) and quintessence ($w_q = w = \mathrm{const}$), and fill the same volume ($R_m = R_q = R$, $y = 1$).

4 Fully clustered quintessence

In the case of fully clustered quintessence, the quintessence field responds to the infall in the same way as matter, and the related continuity equation reads (ML05):

$$\dot{\rho}_q + 3(1+w)\frac{\dot{r}}{r}\rho_q = 0 \quad ; \tag{27}$$

where r is the radial coordinate. An integration from turnaround ($r = R_{\max}$) to a generic configuration $r = R$, using Eq. (11) yields:

$$M_q(R) = M_q(R_{\max})\left(\frac{R}{R_{\max}}\right)^{-3w} \quad ; \tag{28}$$

which shows that the quintessence mass is decreasing with radius, and $\lim_{R\to 0} M_q(R) = 0$, in the case under consideration, $-1 \le w < -1/3$.

Owing to Eqs. (8), (15), and (18), the potential energy of the matter subsystem within the tidal field induced by the quintessence subststem, is:

$$\Omega_m + W_{mq} = -\frac{3}{5}\frac{GM_m^2}{R}\left[1 + (1+3w)m_{qm}\right] \quad ; \tag{29a}$$

$$m_{qm} = \frac{M_q}{M_m} \quad ; \tag{29b}$$

where the mass ratio, m_{qm}, changes with time, according to Eq. (28). At turnaround, $R = R_{\max}$, the assumption of homogeneity makes total energy coincide with potential energy i.e. null kinetic energy (e.g., Horellou & Berge 2005; Percival 2005; ML05).

Owing to Eqs. (8), (15), (16), (19), and (28), the potential energy of the matter subsystem within the tidal field induced by the quintessence subsystem at virialization, is:

$$(\Omega_m + V_{mq})_{\rm vir} = -\frac{3}{5}\frac{GM_m^2}{R_{\max}}\left[\frac{1}{\eta} + \frac{(1+3w)m}{\eta^{1+3w}}\right] \quad ; \tag{30a}$$

$$\eta = \frac{R_{\rm vir}}{R_{\max}} \quad ; \qquad m = (m_{qm})_{\rm vir} = \frac{M_q(R_{\max})}{M_m} \quad ; \tag{30b}$$

where the indices, max and vir, denote turnaround and virialization, respectively.

With regard to the quintessence subsystem, the counterpart of Eqs. (29a), and (30a), via (8), (15), (16), (18), (29b), and (30b) read:

$$\Omega_q + W_{qm} = -\frac{3}{5}\frac{GM_m^2}{R}\left[(1+3w)m_{qm}^2 + m_{qm}\right] \quad ; \tag{31}$$

$$(\Omega_q + V_{qm})_{\rm vir} = -\frac{3}{5}\frac{GM_m^2}{R_{\max}}\left[\frac{(1+3w)m^2}{\eta^{1+6w}} + \frac{m}{\eta^{1+3w}}\right] \quad ; \tag{32}$$

where the assumption of homogeneity makes total energy coincide with potential energy at turnaround.

Using Eqs. (18), (29), and (31), the potential energy of the system at turnaround is:

$$\Omega_{\max} = -\frac{3}{5}\frac{GM_m^2}{R_{\max}}\left[1 + (2+3w)m + (1+3w)m^2\right] \quad ; \tag{33}$$

according to ML05.

Using Eqs. (15), (16), (18), (30), and (32), the potential energy of the system at virialization is:

$$\Omega_{\rm vir} = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\left[\frac{1}{\eta} + \frac{(2+3w)m}{\eta^{1+3w}} + \frac{(1+3w)m^2}{\eta^{1+6w}}\right] \quad ; \tag{34}$$

which is equivalent to:

$$\Omega_{\rm vir} = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\frac{1}{\eta}(1+m\eta^{-3w})\left[1+(1+3w)m\eta^{-3w}\right] \quad ; \tag{35}$$

in terms of radius and quintessence mass at turnaround.

Keeping in mind that, in the case under discussion, the total energy equals the potential energy at turnaround and one half the potential energy at virialization, the requirement of energy conservation (e.g., Wang & Steinhardt 1998; Weinberg & Kamionkonski 2003; Battye & Weller 2003; Horellou & Berge 2005; ML05; but see also Percival 2005) after combination of Eqs. (33) and (35) yields:

$$\eta = \frac{1}{2}\,\frac{1+m\eta^{-3w}}{1+m}\,\frac{1+(1+3w)m\eta^{-3w}}{1+(1+3w)m} \quad ; \tag{36}$$

where the fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, appears on both the left-hand and the right-hand side. The solution of the transcendental Eq. (36) allows the knowledge of the virialized configuration.

The counterpart of Eq. (36) in ML05, Eq. (16) therein, is different for the following reason. The expression, $R\,\partial\Omega_{\rm vir}/\partial R$, has been calculated at constant quintessence mass for an assigned configuration in the current approach, while the quintessence mass, M_q, has been derived with respect to the radius, R, in ML05.

The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, is necessarily restricted within the range, $0 \le \eta \le 1$. In fact, distances cannot be negative and, on the other hand, $\eta > 1$ would contradict the definition of turnaround radius, $R_{\rm max}$. To gain more insight, let us write Eq. (36) under the form:

$$\eta = \frac{1}{2}f(x) = \frac{1}{2}\,\frac{1+x}{1+m}\,\frac{1+(1+3w)x}{1+(1+3w)m} \quad ; \tag{37a}$$

$$x = m\eta^{-3w} \quad ; \qquad 0 \le x \le m \quad ; \qquad 0 \le f(x) \le 2 \quad ; \tag{37b}$$

where the function, $f(x)$, is studied in Appendix B. In the limiting case of a vanishing quintessence field, $m \to 0$, $x \to 0$, and Eq. (37a) reduces to $\eta = 1/2$, which is the known result in absence of dark energy.

The combination of Eqs. (37a) and (37b) yields:

$$\left(\frac{x}{m}\right)^{-1/(3w)} = \frac{1}{2}\,\frac{1+x}{1+m}\,\frac{1+(1+3w)x}{1+(1+3w)m} \quad ; \tag{38}$$

and the virialized configuration is determined by the intersection point between the curve on the left-hand and right-hand side, respectively.

To gain further insight, let us take into consideration a few special cases, namely $w = -1, -2/3, -1/2, -1/3$, the last to be thought of as an interesting limiting situation. The general case, $-1 \le w < -1/3$, is expected to show similar properties as in the closest among the above mentioned particular situations.

In the special case, $w = -1/3$, Eq. (38) reduces to:

$$\frac{x}{m} = \frac{1}{2}\frac{1+x}{1+m} \quad ; \tag{39}$$

the solution of which is:

$$\eta = \frac{x}{m} = \frac{1}{m+2} \quad ; \tag{40}$$

via Eq. (37b). The fractional radius, η, for different values of the fractional mass, m, is represented in Fig. 1, top left.

In the special case, $w = -1/2$, Eq. (38) reduces to:

$$\left(\frac{x}{m}\right)^{2/3} = \frac{1}{2}\frac{2+x-x^2}{2+m-m^2} \quad ; \tag{41}$$

the solution of which is determined by the intersection between curves on the left-hand and right-hand side. The fractional radius, η, for different values of the fractional mass, m, is represented in Fig. 1, top right, where intersections are marked by open squares.

In the special case, $w = -2/3$, Eq. (38) reduces to:

$$\left(\frac{x}{m}\right)^{1/2} = \frac{1}{2}\frac{1-x^2}{1-m^2} \quad ; \tag{42}$$

the solution of which is determined by the intersection between curves on the left-hand and right-hand side. The fractional radius, η, for different

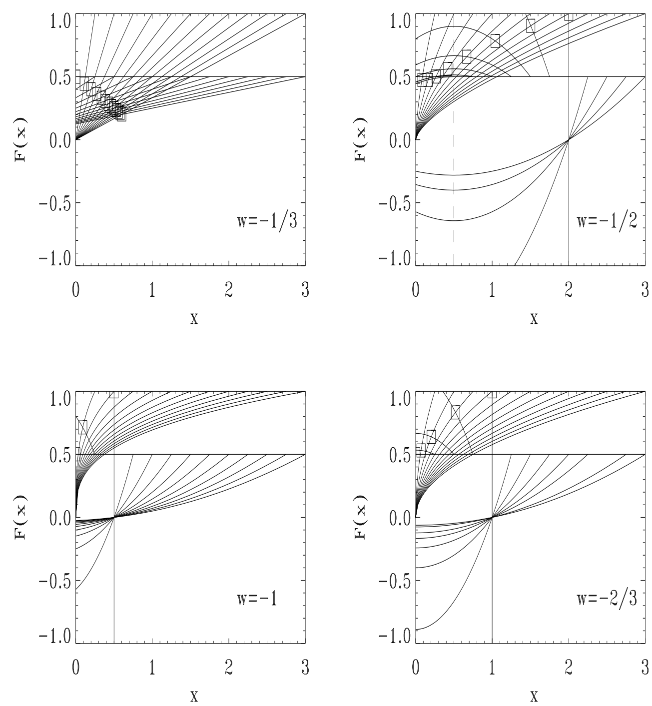

Figure 1: The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, for different values of the fractional mass, $m = M_q(R_{\rm max})/M_m$, with regard to global energy conservation and special choices of the equation of state parameter, w, from top left in the clockwise sense: $-1/3$, $-1/2$, $-2/3$, -1. The value of m related to each case can be read as the abscissa of the ending point of full curves, which occurs at $F(x) = 1/2$. The intersection of the two curves on the left-hand and right-hand side of Eq. (38), related to a virialized configuration, is marked by an open square. Curves corresponding to $F(x) = (x/m)^{-1/(3w)}$ start from the origin. Curves corresponding to $F(x) = (1/2)f(x)$ end at $F(x) = 1/2$, according to Eq. (38). The horisontal axis corresponds to dominant quintessence mass, $m \to +\infty$.

values of the fractional mass, m, is represented in Fig. 1, bottom right, where intersections are marked by open squares.

In the special case, $w = -1$, Eq. (38) reduces to:

$$\left(\frac{x}{m}\right)^{1/3} = \frac{1}{2}\frac{1 - x - 2x^2}{1 - m - 2m^2} \quad ; \tag{43}$$

the solution of which is determined by the intersection between curves on the left-hand and right-hand side. The fractional radius, η, for different values of the fractional mass, m, is represented in Fig. 1, bottom left, where intersections are marked by open squares.

An inspection of Fig. 1 shows the following features.

(i) For assigned turnaround radius, the virialization radius is a decreasing function of the quintessence mass for $w = -1/3$, and an increasing function for $w \le -2/3$. The trend is non monotonic for $w = -1/2$, with the occurrence of a minimum point.

(ii) For assigned equation of state parameter, w, a virialized configuration is allowed for turnaround fractional mass within the range, $0 \le m \le m_0$, where m_0 is the value for which the function on the right-hand side of Eq. (38) is divergent. In the limit, $m \to m_0$, density perturbations turn around and virialize at infinite radius, while density perturbations where $m > m_0$ cannot virialize.

(iii) The threshold fractional mass ranges from $m_0 \to +\infty$ ($w = -1/3$) to $m_0 = 2$ ($w = -1/2$); $m_0 = 1$ ($w = -2/3$); and $m_0 = 1/2$ ($w = -1$). Accordingly, the minimum amount of quintessence within a density perturbation, necessary to prevent matter virialization, is an increasing function of the equation of state parameter, w.

The above results rely on the assumption, that both energy conservation and virialization hold for the whole system (ML05). If, on the other hand, only the matter virializes i.e. the kinetic energy of the quintessence subsystem is null, the procedure should be repeated using Eqs. (29) and (30) instead of (33) and (34). The result is:

$$\eta = \left(\frac{x}{m}\right)^{-1/(3w)} = \frac{1}{2}\frac{1 + (1 + 3w)x}{1 + (1 + 3w)m} \quad ; \tag{44a}$$

$$x = m\eta^{-3w} \quad ; \qquad 0 \le x \le m \quad ; \tag{44b}$$

where the virialized configuration is defined by the intersection of the curves on the left-hand and right-hand side of Eq. (44a), the latter being straight lines.

The counterpart of Eq. (44a) in ML05, Eq. (17) therein, is different for the following reason. Though no indication is provided in ML05 on how Eq. (17) therein has been derived, it can be seen that it is sufficient to omit the terms U_{12} and U_{22} (V_{mq} and Ω_q in the current notation) in the expression of the potential energy, Eq. (4) therein. On the contrary, using Eqs. (29) and (30) implies the omission of V_{qm} and Ω_q, which explains the different results.

In the special case, $w = -1/3$, Eq. (44a) reduces to $\eta = 1/2$, which is the result for matter universes.

In the special case, $w = -1/2$, Eq. (44a) reduces to:

$$\left(\frac{x}{m}\right)^{2/3} = \frac{1}{2}\frac{2-x}{2-m} \quad ; \tag{45}$$

the solution of which is determined by the intersection between curves on the left-hand and right-hand side. The fractional radius, η, for different values of the fractional mass, m, is represented in Fig. 2, top right, where intersections are marked by open squares.

In the special case, $w = -2/3$, Eq. (44a) reduces to:

$$\left(\frac{x}{m}\right)^{1/2} = \frac{1}{2}\frac{1-x}{1-m} \quad ; \tag{46}$$

the solution of which is determined by the intersection between curves on the left-hand and right-hand side. The fractional radius, η, for different values of the fractional mass, m, is represented in Fig. 2, bottom right, where intersections are marked by open squares.

In the special case, $w = -1$, Eq. (44a) reduces to:

$$\left(\frac{x}{m}\right)^{1/3} = \frac{1}{2}\frac{1-2x}{1-2m} \quad ; \tag{47}$$

the solution of which is determined by the intersection between curves on the left-hand and right-hand side. The fractional radius, η, for different values of the fractional mass, m, is represented in Fig. 2, bottom left, where intersections are marked by open squares.

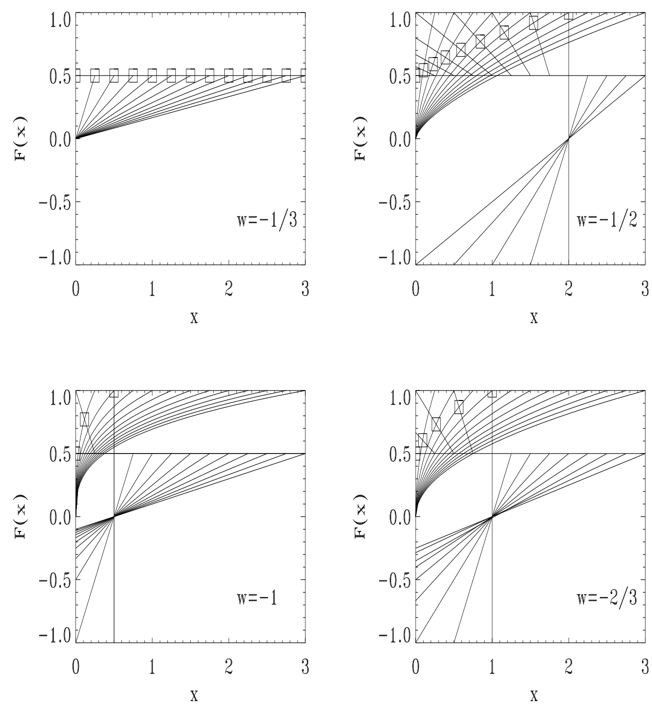

Figure 2: The fractional radius, $\eta = R_{\mathrm{vir}}/R_{\mathrm{max}}$, for different values of the fractional mass, $m = M_q(R_{\mathrm{max}})/M_m$, with regard to partial energy conservation and special choices of the equation of state parameter, w, from top left in the clockwise sense: $-1/3$, $-1/2$, $-2/3$, -1. The value of m related to each case can be read as the abscissa of the ending point of full curves, which occurs at $F(x) = 1/2$. The intersection of the two curves on the left-hand and right-hand side of Eq. (44a), related to a virialized configuration, is marked by an open square. Curves corresponding to $F(x) = (x/m)^{-1/(3w)}$ start from the origin. Straight lines corresponding to $F(x) = (1/2)[1+(1+3w)x]/[1+(1+3w)m]$ end at $F(x) = 1/2$, according to Eq. (44a). The horisontal axis corresponds to dominant quintessence mass, $m \to +\infty$.

An inspection of Fig. 2 shows similar features as in Fig. 1 (see above), with the following difference. For an assigned turnaround radius, the virialization radius is an increasing function of the quintessence mass for $w < -1/3$, and independent of the quintessence mass (as in matter universes) for $w = -1/3$. Accordingly, a monotonic trend is exhibited.

The results plotted in Fig. 1 and Fig. 2 imply the assumption of energy conservation with regard to the whole system and to the matter subsystem, respectively, the latter requirement being more restrictive than the former. For a more detailed discussion on energy conservation in density perturbations with matter and dark energy, see ML05 and Percival (2005).

The comparison of the current results (Figs. 1-2) with their counterparts in ML05 (Fig. 2 therein, related to $w = -4/5$) shows that the trend of the fractional radius, η, as a function of the fractional mass, m, is increasing in both cases with regard to the whole system, while is increasing in the current paper and decreasing in ML05 with regard to the matter subsystem. A closer agreement could occur within the range, $-1/2 \le w < -1/3$. The reasons of the above mentioned discrepancy are due to different assumptions made in the current paper and in ML05, as explained above.

5 Partially clustered quintessence

In the case of partially clustered quintessence, the quintessence field responds to the infall to a lesser extent with respect to the matter. In the limiting situation of unclustered quintessence, the related continuity equation reads (ML05):

$$\dot{\rho}_q + 3(1+w)\frac{\dot{a}}{a}\rho_q = 0 \quad ; \tag{48}$$

where a is the cosmological scale factor, which is related to the redshift, z, via $a \propto (1+z)^{-1}$. In the special case, $w = -1$, Eqs. (27) and (48) coincide, yielding $\rho_q = \mathrm{const}$ or, in other terms, a cosmological constant.

An integration from turnaround ($r = R_{\max}$) to a generic configuration $r = R$, using Eq. (11) yields:

$$M_q(R) = M_q(R_{\max})\left(\frac{R}{R_{\max}}\right)^3\left(\frac{a}{a_{\max}}\right)^{-3(1+w)} \quad ; \tag{49a}$$

$$\frac{a}{a_{\max}} = \frac{1+z_{\max}}{1+z} \quad ; \tag{49b}$$

which shows that the quintessence mass is decreasing with radius, and $\lim_{R\to 0} M_q(R) = 0$, in the case under consideration, $-1 \le w < -1/3$.

In the general case of partially clustered quintessence, it may safely be expected that the virialized configuration of matter subsystem lies between the limiting situation of fully clustered and unclustered quintessence, respectively. It is worth remembering that energy conservation can no longer be assumed, as in presence of unclustered quintessence (e.g., Horellou & Berge 2005; Percival 2005; ML05). Without loss of generality, let us suppose that a specified virialized configuration is attained along the following steps: (i) from turnaround to virialization related to fully clustered quintessence, and (ii) from virialization related to fully clustered quintessence to virialization related to partially clustered quintessence.

To perform the transition between the above mentioned steps, the quintessence mass must instantaneously change from $M_q(R'_{\rm vir})$ to $M_q(R_{\rm vir})$, where the presence and the absence of the prime denotes the virialized configuration related to fully or partially clustered quintessence, respectively.

The changes in self, tidal, and interaction potential energy, keeping in mind Eqs. (8), (15), and (16), are:

$$\Delta\Omega_m = -\frac{3}{5}\frac{GM_m^2}{R'_{\rm vir}} + \frac{3}{5}\frac{GM_m^2}{R'_{\rm vir}} \quad ; \tag{50}$$

$$\Delta V_{mq} = \Delta W_{mq} = -\frac{3}{5}(1+3w)\frac{GM_m M_q(R'_{\rm vir})}{R'_{\rm vir}} + \frac{3}{5}(1+3w)\frac{GM_m M_q(R_{\rm vir})}{R'_{\rm vir}} \quad ; \tag{51}$$

$$\Delta V_{qm} = \Delta W_{qm} = -\frac{3}{5}\frac{GM_m M_q(R'_{\rm vir})}{R'_{\rm vir}} + \frac{3}{5}\frac{GM_m M_q(R_{\rm vir})}{R'_{\rm vir}} \quad ; \tag{52}$$

$$\Delta\Omega_q = -\frac{3}{5}(1+3w)\frac{GM_q^2(R'_{\rm vir})}{R'_{\rm vir}} + \frac{3}{5}(1+3w)\frac{GM_q^2(R_{\rm vir})}{R'_{\rm vir}} \quad ; \tag{53}$$

from which the changes in total energy within the matter subsystem and the whole system, can be deduced.

To this aim, let us define the fractional masses and the fractional radii:

$$\mu = \frac{M_q(R_{\rm vir})}{M_q(R_{\rm max})} \quad ; \qquad \Delta\mu = \frac{M_q(R'_{\rm vir}) - M_q(R_{\rm vir})}{M_q(R_{\rm max})} \quad ; \tag{54}$$

$$\eta = \frac{R_{\rm vir}}{R_{\rm max}} \quad ; \qquad \Delta\eta = \frac{R'_{\rm vir} - R_{\rm vir}}{R_{\rm max}} \quad ; \tag{55}$$

and the combination of Eqs. (28), (54), and (55) yields:

$$\mu + \Delta\mu = \frac{M_q(R'_{\rm vir})}{M_q(R_{\rm max})} = \left(\frac{R'_{\rm vir}}{R_{\rm max}}\right)^{-3w} = \eta_{\rm FC}^{-3w} \quad ; \tag{56}$$

$$\eta + \Delta\eta = \frac{R'_{\rm vir}}{R_{\rm max}} = \eta_{\rm FC} \quad ; \tag{57}$$

in the special case of fully clustered quintessence, $\Delta\mu = 0$, the results of Sect. 4 continue to hold provided η is replaced by $\eta_{\rm FC}$ therein.

The substitution of Eqs. (30) and (54)-(57) into (50)-(53) produces:

$$\Delta\Omega_m = 0 \quad ; \tag{58}$$

$$\Delta V_{mq} = \Delta W_{mq} = -\frac{3}{5}(1+3w)\frac{GM_m^2}{R'_{\rm vir}} m\Delta\mu \quad ; \tag{59}$$

$$\Delta V_{qm} = \Delta W_{qm} = -\frac{3}{5}\frac{GM_m^2}{R'_{\rm vir}} m\Delta\mu \quad ; \tag{60}$$

$$\Delta\Omega_q = -\frac{3}{5}(1+3w)\frac{GM_m^2}{R'_{\rm vir}} m^2[2\mu\Delta\mu + (\Delta\mu)^2] \quad ; \tag{61}$$

in terms of virialized configurations related to fully clustered quintessence and to the change, $\Delta\mu$.

At this stage, the virialized configuration of the matter subsystem can be determined using a similar procedure with respect to the special case of fully clustered quintessence. The change in total energy, related to the transition from virialized configurations where the quintessence is fully clustered ($R = R'_{\rm vir}$) to virialized configurations where the quintessence is partially clustered ($R = R_{\rm vir}$), reads:

$$\frac{1}{2}\Omega(R'_{\rm vir}) + \Delta\Omega = \frac{1}{2}\Omega(R_{\rm vir}) \quad ; \tag{62a}$$

$$\Delta\Omega = \Delta\Omega_m + \Delta V_{mq} + \Delta V_{qm} + \Delta\Omega_q \quad ; \tag{62b}$$

and the combination of Eqs. (34), (54)-(57), and (58)-(62) yields:

$$\Omega(R'_{\rm vir}) = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\left[\frac{1}{\eta_{\rm FC}} + \frac{(2+3w)m}{\eta_{\rm FC}^{1+3w}} + \frac{(1+3w)m^2}{\eta_{\rm FC}^{1+6w}}\right] \quad ; \tag{63}$$

$$\Omega(R_{\rm vir}) = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\left[\frac{1}{\eta} + \frac{(2+3w)m\mu}{\eta} + \frac{(1+3w)m^2\mu^2}{\eta}\right] \quad ; \tag{64}$$

$$\Delta\Omega = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\left[(2+3w)m + (1+3w)m^2(2\mu+\Delta\mu)\right]\frac{\Delta\mu}{\eta_{\rm FC}} \quad ; \tag{65}$$

in terms of radius and quintessence mass at turnaround.

Keeping in mind that, in the special case of fully clustered quintessence, the total energy equalizes the potential energy at turnaround and one half the potential energy at virialization, the substitution of Eqs. (33), (64), and (65) into (62a) produces:

$$[1+(2+3w)m+(1+3w)m^2] + [(2+3w)m + (1+3w)m^2(2\eta_{\rm FC}^{-3w} - \Delta\mu)]\frac{\Delta\mu}{\eta_{\rm FC}}$$
$$= \frac{1}{2}[1+(2+3w)m\mu + (1+3w)m^2\mu^2]\frac{1}{\eta} \quad ; \tag{66}$$

where, using Eq. (56) and performing some algebra, the term within brackets on the right-hand side of Eq. (66) may be cast into the form:

$$1+(2+3w)m\mu+(1+3w)m^2\mu^2 =$$
$$1+(2+3w)m\eta_{\rm FC}^{-3w} + (1+3w)m^2\eta_{\rm FC}^{-6w} - \phi(w,m,\eta_{\rm FC},\Delta\mu) \quad ; \tag{67a}$$

$$\phi(w,m,\eta_{\rm FC},\Delta\mu) =$$
$$m[(2+3w) + 2(1+3w)m\eta_{\rm FC}^{-3w} - (1+3w)m\Delta\mu]\Delta\mu \quad ; \tag{67b}$$

and the combination of Eqs. (66) and (67) yields:

$$\eta = \frac{1}{2}\frac{[1+(2+3w)m\eta_{\rm FC}^{-3w} + (1+3w)m^2\eta_{\rm FC}^{-6w}] - \phi}{[1+(2+3w)m+(1+3w)m^2] + \eta_{\rm FC}^{-1}\phi} \quad ; \tag{68}$$

which depends on the parameters, w, m, $\eta_{\rm FC}$, and $\Delta\mu$.

In the special case of fully clustered quintessence, $\Delta\mu = 0$ i.e. $\mu = \eta_{\rm FC}^{-3w}$, and Eq. (67) reduces to (36) where η has to be replaced by $\eta_{\rm FC}$. The solution of the transcendental Eq. (68) allows the knowledge of the virialized configuration for fixed $\Delta\mu$.

The counterpart of Eq. (68) in ML05, Eq. (23) therein, is different for reasons discussed in Sect. 4 and, in addition, due to a different choice of the parameter related to partial clustering, γ therein instead of $\Delta\mu$ or $\Delta\eta$, according to Eq. (56).

The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, is necessarily restricted within the range, $0 \le \eta \le 1$. In fact, distances cannot be negative and, on the other hand, $\eta > 1$ would contradict the definition of turnaround radius, $R_{\rm max}$. In addition, the particularization of Eqs. (28) and (49) to the related virialized configurations, shows that $M_q(R_{\rm vir}) \le M_q(R'_{\rm vir})$ within the range of interest, $-1 \le w < -1/3$, which implies $\Delta\mu \ge 0$ via Eq. (54). Accordingly, $0 \le \Delta\mu \le \eta_{\rm FC}^{-3w}$, and the sign of ϕ is opposite to the sign of $\Delta\Omega$ via Eq. (65) which, in turn, is equal to the sign of $\eta - \eta_{\rm FC}$.

To gain more insight, let us express the increment, $\Delta\mu$, in terms of the variable, $\eta_{\rm FC}$, and a degree of quintessence de-clustering, ζ, as:

$$\Delta\mu = \zeta\eta_{\rm FC}^{-3w} \quad ; \qquad 0 \le \zeta \le 1 \quad ; \tag{69}$$

the substitution of Eq. (69) into (67b) yields:

$$\phi(w, m, \eta_{\rm FC}, \Delta\mu) = (2 + 3w)m\zeta\eta_{\rm FC}^{-3w} + (1 + 3w)m^2\zeta(2 - \zeta)\eta_{\rm FC}^{-6w} \quad ; \tag{70}$$

where the effect of quintessence partial clustering is expressed by the parameter, ζ, where the limit of fully clustered and fully de-clustered quintessence corresponds to $\zeta = 0$ and $\zeta = 1$, respectively. The latter, of course, has no physical meaning, as quintessence cannot be devoided from the volume filled by the matter subsystem. On the other hand, it makes an upper limit to the quintessence de-clustering parameter, ζ. Further details on the function, ϕ, are given in Appendix C.

The above results may be summarized as follows. Given a density perturbation with assigned value of quintessence equation of state parameter, w, $-1 \le w < -1/3$, quintessence to matter mass ratio at turnaround, m, virialized to turnaround size ratio in the special case of fully clustered quintessence, $\eta_{\rm FC}$, $0 \le \eta_{\rm FC} \le 1$, and degree of quintessence de-clustering, ζ, $0 \le \zeta \le 1$, the related virialized to turnaround size ratio in the case of partially clustered quintessence, η, is expressed by Eqs. (68), (69), and (70).

Let us repeat that the special case of fully de-clustered quintessence, $\zeta = 1$, makes a convenient upper limit but, on the other hand, it has little physical meaning. A true upper limit is related to the special case of unclustered quintessence, $\zeta = \zeta_{\rm unc}$, where the combination of Eqs. (49), (54), (55), (56), and (69) yields:

$$\eta = (1 - \zeta_{\rm unc})^{1/3} \eta_{\rm FC}^{-w} \left(\frac{1 + z_{\rm max}}{1 + z} \right)^{1+w} \quad ; \tag{71}$$

which, in addition to Eq. (68), makes a further constraint for determining the virialized configuration, with regard to selected turnaround and virialization epoch.

To get further insight, let us express the fractional radius, η, in terms of the parameter, $x = m\eta_{\rm FC}^{-3w}$. Accordingly, Eq. (68) reads:

$$\eta = \frac{1}{2} \frac{[1 + (2 + 3w)x + (1 + 3w)x^2] - \phi(x; w, \zeta)}{[1 + (2 + 3w)m + (1 + 3w)m^2] + (x/m)^{1/(3w)} \phi(x; w, \zeta)} \quad ; \tag{72}$$

where the function, $\phi(x; w, \zeta)$, is expressed by Eq. (114), Appendix C.

The dependence, $\eta = \eta(m)$, is represented in Fig. 3 with regard to same cases as in Fig. 1, but different degrees of quintessence de-clustering, $\zeta = 0$ (squares), 0.25 (triangles), 0.5 (asterisks), 0.75 (crosses), and 1 (diamonds). The starting point on the left, (0,0.5), marks the limiting situation of vanishing quintessence, and necessarily coincides in all cases. For sufficiently high values of quintessence equation of state parameter, $w \le -1/3$, lower curves correspond to less clustered quintessence and vice versa. For sufficiently low values of quintessence equation of state parameter, $-1 \le w \le -2/3$, lower curves correspond to more clustered quintessence and vice versa. For intermediate values of quintessence equation of state parameter, $w \approx -1/2$, different curves intersect one with the other, but a minimum point occurs in the whole range between fully clustered ($\zeta = 0$) and fully de-clustered ($\zeta = 1$) quintessence. In any case, the general trend remains unchanged with respect to the limiting situation of fully clustered quintessence.

The above results rely on the assumption, that both energy conservation and virialization hold for the whole system (ML05). If, on the other hand, only the matter virializes i.e. the kinetic energy of the quintessence system is null, the procedure should be repeated with regard to the matter subsystem only. Accordingly, Eqs. (62), (63), (64), and (65) are turned into:

$$E_m(R'_{\rm vir}) + \Delta E_m = E_m(R_{\rm vir}) \quad ; \tag{73}$$

$$\Delta E_m = \Delta\Omega_m + \Delta V_{mq} \quad ; \tag{74}$$

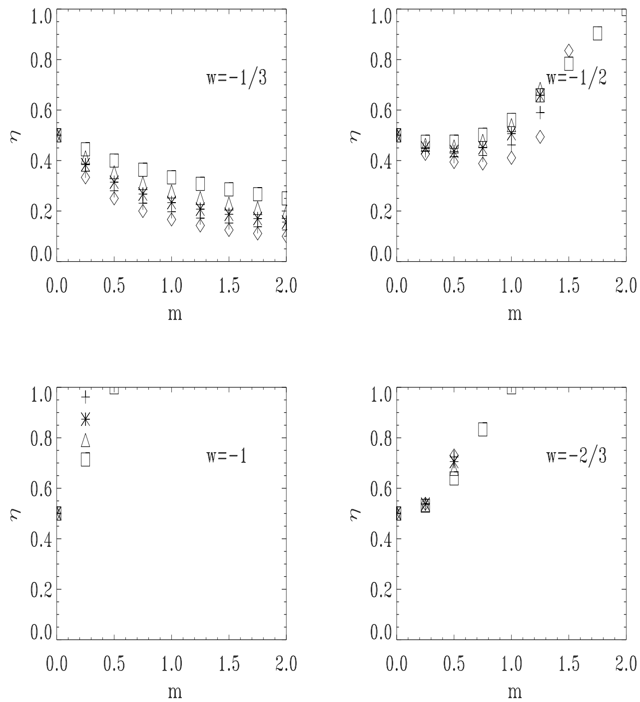

Figure 3: The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, with regard to global energy conservation, for different values of the fractional mass, $m = M_q(R_{\rm max})/M_m$, the quintessence de-clustering parameter, ζ, and the quintessence equation of state parameter, w. From top left in the clockwise sense: $w = -1/3$, $-1/2$, $-2/3$, -1. Caption of symbols: squares, $\zeta = 0$ (fully clustered quintessence, as in Fig. 1); triangles, $\zeta = 1/4$; asterisks, $\zeta = 1/2$; crosses, $\zeta = 3/4$; diamonds, $\zeta = 1$ (fully de-clustered quintessence). The value of m related to each symbol, starting from the left, is $m = i/4$, $i = 0, 1, 2, ...$, and the special case, $i = 0$, corresponds to a vanishing quintessence.

$$2E_m(R'_{\rm vir}) = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\frac{1}{\eta_{\rm FC}}\left[1 + \frac{(1+3w)m}{\eta_{\rm FC}^{3w}}\right] \quad ; \tag{75}$$

$$2E_m(R_{\rm vir}) = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\frac{1}{\eta}[1 + (1+3w)m\mu] \quad ; \tag{76}$$

$$\Delta E_m = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}(1+3w)\frac{m\Delta\mu}{\eta_{\rm FC}} \quad ; \tag{77}$$

where $E_m = (\Omega_m + W_{mq})/2 = (\Omega_m + V_{mq})/2$ is the total energy of the virialized matter subsystem, and ΔE_m is the energy change due to the transition from a virialized configuration where the quintessence is fully clustered, to its counterpart where the quintessence is partially clustered. Using energy conservation in the former alternative, the substitution of Eqs. (75), (76), and (77) into (73) yields after some algebra:

$$\eta = \frac{1}{2}\frac{[1 + (1+3w)m\eta_{\rm FC}^{-3w}] - \phi}{[1 + (1+3w)m] + \eta_{\rm FC}^{-1}\phi} \quad ; \tag{78}$$

$$\phi(w, m, \zeta) = (1+3w)m\zeta\eta_{\rm FC}^{-3w} \quad ; \tag{79}$$

where Eq. (56) has also been used.

In terms of the parameter, $x = m\eta_{\rm FC}^{-3w}$, Eqs. (78) and (79), translate into:

$$\eta = \frac{1}{2}\frac{[1 + (1+3w)x] - \phi}{[1 + (1+3w)m] + (x/m)^{1/(3w)}\phi} \quad ; \tag{80}$$

$$\phi(x, w, \zeta) = (1+3w)\zeta x \quad ; \tag{81}$$

in the limit of fully clustered quintessence, $\zeta = 0$, $\phi = 0$, Eq. (80) coincides with Eq. (44a).

The dependence, $\eta = \eta(m)$, is represented in Fig. 4 with regard to the same cases as in Fig. 2, but different degrees of quintessence de-clustering, $\zeta = 0$ (squares), 0.25 (triangles), 0.5 (asterisks, 0.75 (crosses), and 1 (diamonds). The starting point on the left, $(0, 0.5)$, marks the limiting situation of vanishing quintessence, and necessarily coincides in all cases. It is apparent that lower curves correspond to more clustered quintessence and vice versa. In the limit, $w \to -1/3$, virialized configurations coincide with their counterparts in absence of quintessence, independent of the degree of clustering. On the other hand, the dependence is enhanced as the quintessence equation of state parameter, w, attains lower values up to -1. In any case,

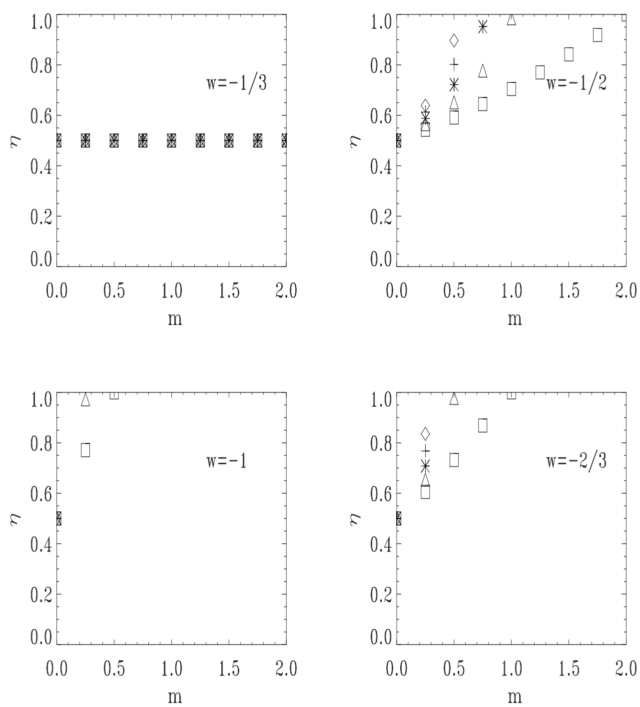

Figure 4: The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, for different values of the fractional mass, $m = M_q(R_{\rm max})/M_m$, the quintessence de-clustering parameter, ζ, and the quintessence equation of state parameter, w, with regard to partial energy conservation. Captions as in Fig. 3.

the general trend remains unchanged with respect to the special situation of fully clustered quintessence.

In dealing with fully clustered quintessence, an inspection of Figs. 3 and 4 shows that partial energy conservation yields larger virialized configurations, with respect to their counterparts where global energy conservation holds. An opposite trend is found in ML05, for $w = -4/5$, in the case of fully clustered (Fig. 2 therein) and unclustered (Fig. 4 therein) quintessence. The reasons of the above mentioned discrepancy are due to different assumptions made in the current paper and in ML05, as explained in Sect. 4.

The dependence, $\eta = \eta(\zeta)$, is represented in Fig. 5 with regard to both global (squares) and partial (diamonds) energy conservation (in the special case of fully clustered quintessence), for $m = 1/4$ and $w = -1/3, -1/2, -2/3, -1$. In both cases, larger fractional radii are attained for (algebraically) lower quintessence equation of state parameters, w. In addition, with regard to fully clustered quintessence, partial energy conservation yields larger virialized configurations when compared to their counterparts where global energy conservation holds. Accordingly, partial energy conservation implies a larger amount of kinetic energy in matter subsystems, with respect to global energy conservation. An opposite result has been found in ML05 (Fig. 3 therein) for $w = -4/5$ and $m = 1/5$. The reasons of the above mentioned discrepancy are due to different assumptions made in the current paper and in ML05, as explained in Sect. 4.

The dependence, $\eta = \eta(w)$, is represented in Fig. 6 with regard to both global (squares) and partial (diamonds) energy conservation (in the special case of fully clustered quintessence), for $m = 1/4$ and $\zeta = 0$ (fully clustered quintessence), 1/4, 3/4, 1 (fully de-clustered quintessence). In both cases, the trend remains unchanged as the degree of de-clustering gets increased. Other features have already been discussed above. A different result has been found in ML05 (Fig. 5 therein), for $w = -4/5$ and $m = 1/5$. The reasons of this discrepancy are due to different assumptions made in the current paper and in ML05, as explained in Sect. 4.

6 Discussion.

In absence of a full knowledge on the dark energy, an investigation on the virialization of spherical overdensities appears to be physically meaningful, even

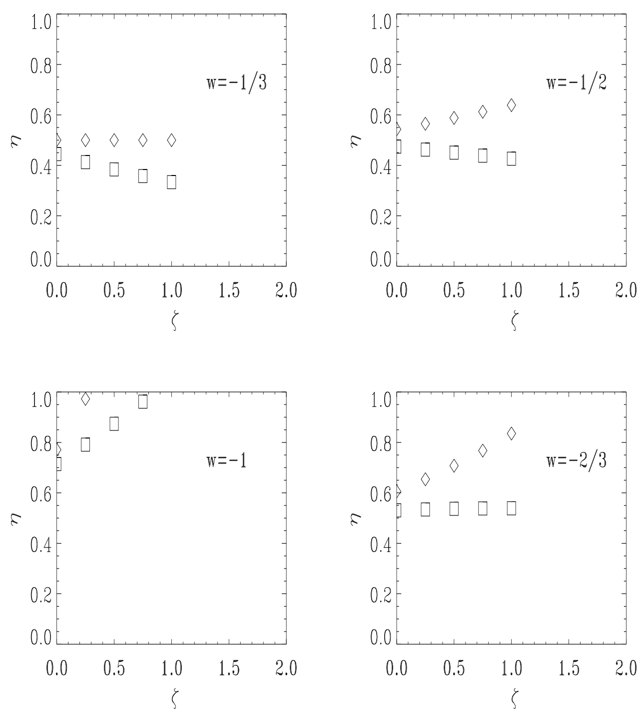

Figure 5: The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, as a function of the quintessence de-clustering parameter, ζ, for a fractional mass, $m = M_q(R_{\rm max})/M_m$, and values of the quintessence equation of state parameter, w, from top left in the clockwise sense: $-1/3$, $-1/2$, $-2/3$, -1. Global and partial energy conservation (in the special case of fully clustered quintessence) correspond to squares and diamonds, respectively.

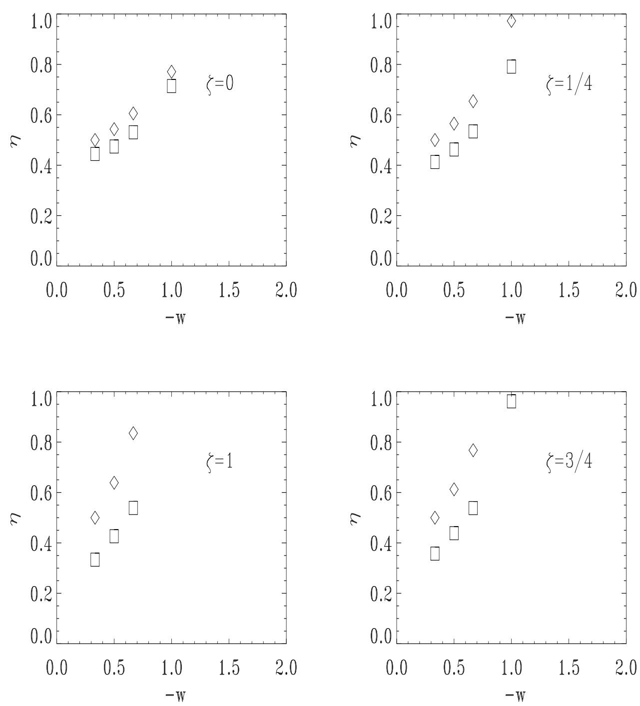

Figure 6: The fractional radius, $\eta = R_{\rm vir}/R_{\rm max}$, as a function of the quintessence equation of state parameter, w, for a fractional mass, $m = 1/4$, and values of the quintessence de-clustering parameter, ζ, from top left in the clockwise sense: 0 (fully clustered quintessence), 1/4, 3/4, 1 (fully de-clustered quintessence). Global and partial energy conservation (in the special case of fully clustered quintessence) correspond to squares and diamonds, respectively.

if particular assumptions are made. More specifically, the potential induced by quintessence has been conceived as due to quintessence "particles", interacting via an inverse square distance law, with strength equal to $(1+3w)G$, $-1 \le w < -1/3$. Accordingly, the potential induced by a quintessence particle scales as an inverse distance, $\mathcal{V}(r_p) \propto 1/r_p$, which makes the formulation of the virial theorem, $2T = \kappa\Omega$, $\mathcal{V}(r_p) \propto r_p^\kappa$, $\kappa = -1$ (e.g., Landau & Lifchitz 1966, Chap. II, § 10).

With regard to the transition from turnaround to virialization, the reference case of energy conservation may be related to either the whole (quintessence + matter) system, or the matter subsystem within the tidal field induced by the quintessence subsystem. According to ML05, global energy conservation occurs in the special case of fully clustered quintessence. The same is assumed with regard to partial energy conservation, even if it is not clearly specified in ML05 [Eq. (17) therein].

Bearing in mind the reference cases of global and partial energy conservtion, different degrees of quintessence clustering may be expressed via the de-clustering parameter, ζ, $0 \le \zeta \le 1$, which is related to the change in mass with respect to a selected reference case, according to Eqs. (54)-(57) and (69). The limiting cases, $\zeta = 0, 1$, represent fully clustered and fully de-clustered quintessence, respectively.

Although typical values of the fractional mass at turnaround, $m = M_q(T_{\max})/M_m$, do not exceed a value of about 0.9 (e.g., ML05), the analysis of a larger range shows interesting features, namely (i) the existence of a threshold in m, increasing with the quintessence equation of state parameter, w, above which no virialized configuration is allowed, see Figs. 3 and 4, and (ii) the occurrence of a non monotonic trend in the fractional radius, η, as a function of the fractional mass, m, with the occurrence of a minimum, for $w = -1/2$, a decreasing monotonic trend for $w = -1/3$, and an increasing monotonic trend for $w = -2/3, -1$, with regard to global energy conservation in the limit of fully clustered quintessence, see Fig. 3.

The dependence, $\eta = \eta(\zeta)$, related to $m = 1/4$, is also monotonic for both global and partial energy conservation in the limiting case of fully clustered quintessence, see Fig. 5. Owing to Eq. (71), the special case of unclustered quintessence occurs within the whole range, $0 \le \zeta < 1$, according if $1 \le a_{\mathrm{vir}}/a_{\max} < 1500$ and/or $-1 \le w < -1/3$.

The dependence, $\eta = \eta(w)$, shows a monotonic trend for both global and partial energy conservation in the limiting case of fully clustered quintessence, see Fig. 6, with increasingly different trends as ζ increases. It can be seen that virialized configurations cannot take place for both sufficiently low w and sufficiently large ζ.

In any case, more extended virialized configurations occur for partial (with respect to global) energy conservation in the limiting case of fully clustered quintessence, see Figs. 5 and 6. Accordingly, the latter alternative yields bounder configurations with respect to the former. In addition, an increasing degree of quintessence de-clustering makes more extended virialized configurations provided partial energy conservation holds in the special case of fully clustered quintessence, see Fig. 5. When global energy conservation holds in the special case of fully clustered quintessence, a similar trend occurs for sufficiently low quintessence equation of state parameters, $w < -2/3$. No change appears for $w = -2/3$, while the trend is reversed for $w > -2/3$.

The above results are different from their analogon in ML05, even if the expression of the potential energy is the same as in the current paper. The discrepancy could be due to (i) different formulations of the virial theorem, as discussed in Sect. 4 and in Appendix A, with regard to both the whole system and the matter subsystem, and (ii) different descriptions of de-clustered quintessence, by defining a quintessence de-clustering parameter in the current paper, and modifying the quintessence continuity equation in ML05, Eqs. (9)-(11) therein.

If the fractional radius, η, can be deduced from observations, it (together with the related uncertainty) translates into the plane of Figs. 3-6 as a horizontal band, and the value of the fractional mass, m, the degree of quintessence de-clustering, ζ, and the quintessence equation of state parameter, w, can be constrained. In addition, global and partial energy conservation in the special case of fully clustered quintessence, can be tested.

7 Conclusion

The virialization of matter overdensities within dark energy subsystems has been considered under a number of simplifying assumptions, namely (i) spherical-symmetric density profiles, (ii) time-independent quintessence equation of state parameter, w, and (iii) sole gravitational interaction between dark energy scalar field and matter. The quintessence subsystem has been conceived as made of "particles" whose mutual interaction has intensity equal to $(1+3w)G$ and scales as the inverse square of the distance. The related expression of self and tidal potential energy and formulation of the virial theorem for subsystems, have been found to be consistent with their matter counterparts, passing from $-1 \le w < -1/3$ to $w = 0$. In the special case of fully clustered quintessence, following ML05, it has been assumed either global energy conservation related to the whole system, or partial energy conservation related to the matter subsystem within the tidal field induced by the quintessence subsystem. Further investigation has been devoted to a few special cases, namely a limiting situation, $w = -1/3$, and lower values, $w = -1/2, -2/3, -1$, where the last one mimics the presence of a cosmological constant.

The special case of fully clustered (i.e. collapsing together with the matter) quintessence has been studied in detail, following a similar procedure as in ML05. The general case of partially clustered quintessence has been considered in terms of a degree of quintessence de-clustering, ζ, $0 \le \zeta \le 1$, ranging from fully clustered ($\zeta = 0$) to completely de-clustered ($\zeta = 1$) quintessence, respectively. The special case of unclustered (i.e. remaining homogeneous) quintessence has also been discussed.

The trend exhibited by the fractional radius, η, as a function of the fractional mass, m, the degree of quintessence de-clustering, ζ, and the quintessence equation of state parameter, w, has been found to be different from its counterpart reported in earlier attempts (e.g., ML05). In particular, no clear dichotomy with respect to the limiting situation of a vanishing quintessence, $\eta = 1/2$, has been shown when global or partial energy conservation holds in the special case of fully clustered quintessence, with $\eta > 1/2$ preferred. The reasons of the above mentioned discrepancy have been recognized as owing to (i) different formulations of the virial theorem, and (ii) different descriptions of de-clustered quintessence, with respect to the case of fully clustered quintessence.

References

[1] Amendola, L. 2000, PhRvD 62, 043511

[2] Battye, R.A., Weller, J. 2003, PhRvD 68, 083506

[3] Brosche, P., Caimmi, R., Secco, L. 1983, A&A 125, 338

[4] Caimmi, R. 2003, AN 324, 250

[5] Caimmi, R., Secco, L., Brosche, P. 1984, A&A 139, 411

[6] Caimmi, R., Secco, L. 1992, ApJ 395, 119

[7] Caldwell, R.R., Dave, R., Steinhardt, P.J. 1988a, ApSS 261, 303

[8] Caldwell, R.R., Dave, R., Steinhardt, P.J. 1988b, PhRvL 80, 1582

[9] Chandrasekhar, S. 1969, Ellipsoidal Figures of Equilibrium, Yale University Press, New Haven

[10] Gunn, J.E., Gott, J.R., III 1972, ApJ 176, 1

[11] Horellou, C., Berge, J. 2005, MNRAS 360, 1393

[12] Iliev, I.T., Shapiro, P.R. 2001, MNRAS 325, 468

[13] Lahav, O., Lilje, P.B., Primack, J.R., Rees, M.J. 1991, MNRAS 251, 128

[14] Landau, L., Lifchitz, E. 1966 *Mecanique*, Mir, Moscow

[15] Limber, D.N. 1959, ApJ 130, 414

[16] MacMillan, D.W. 1930 *The Theory of the Potential*, Dover Publications, Inc. New York, 1958

[17] Maor, I., Lahav, O. 2005, JCAP 7, 3 (ML05)

[18] Mota, D.F., van de Bruck, C. 2004, A&A 421, 71

[19] Peebles, P.J.E. Ratra, B. 1988, ApJ 325, 17

[20] Peebles, P.J.E. Ratra, B. 2003, Rev. Mod. Phys. 75, 559

[21] Percival, W.J. 2005, A&A 443, 819

[22] Ratra, B. 1988, Peebles, P.J.E. 1988, PhRvD 37, 3406

[23] Rubiño-Martin, J.A., Rebolo, R., Carreira, P., et al. 2003, MNRAS 341, 1084

[24] Sievers, J.L., Bond, J.R., Cartwright, J.K., et al. 2003, ApJ 591, 599

[25] Spergel, D.N., Verde, L., Peiris, H.V., et al. 2003, ApJS 148, 175

[26] Wang, L., Steinhardt, P.J. 1988, ApJ 508, 483

[27] Wang, L. 2006, ApJ 640, 18

[28] Weinberg, N.N., Kamionkonsky M. 2003, MNRAS 341, 251

[29] Wetterich, C. 1988, Nucl. Phys. B 302, 668

[30] Wetterich, C. 1995, A&A 301, 321

Appendix

A. On the virial theorem for subsystems

Let i and j denote concentric, spherical-symmetric, homogeneous subsystems, R_i and R_j, $R_i \le R_j$, the related radii, and let the potential and the potential energy terms be expressed by Eqs. (2), (8), (9), (10), and (14). The derivation of the potential energy terms with respect to the inner or outer radius yields:

$$R_i \frac{\partial \Omega_i}{\partial R_i} = -\Omega_i \quad ; \qquad R_j \frac{\partial \Omega_i}{\partial R_j} = 0 \quad ; \tag{82}$$

$$R_j \frac{\partial \Omega_j}{\partial R_j} = -\Omega_j \quad ; \qquad R_i \frac{\partial \Omega_j}{\partial R_i} = 0 \quad ; \tag{83}$$

$$\frac{R_i}{1+3w_j} \frac{\partial W_{ij}}{\partial R_i} = \frac{R_i}{1+3w_i} \frac{\partial W_{ji}}{\partial R_i} = -\frac{1}{2} \frac{V_{ij}}{1+3w_j} \quad ; \tag{84}$$

$$\frac{R_j}{1+3w_i} \frac{\partial W_{ji}}{\partial R_j} = \frac{R_j}{1+3w_j} \frac{\partial W_{ij}}{\partial R_j} = -\frac{V_{ij}}{1+3w_j} \left(\frac{5}{4} y^2 - \frac{3}{4} \right) \quad ; \tag{85}$$

$$R_i \frac{\partial V_{ij}}{\partial R_i} = 2V_{ij} \quad ; \qquad R_j \frac{\partial V_{ij}}{\partial R_j} = -3V_{ij} \quad ; \tag{86}$$

$$R_j \frac{\partial V_{ji}}{\partial R_j} = -V_{ij} \left[\left(\frac{5}{4} y^2 - \frac{15}{4} \right) + \left(\frac{5}{4} y^2 - \frac{3}{4} \right) \frac{1+3w_i}{1+3w_j} \right] \quad ; \tag{87}$$

$$R_i \frac{\partial V_{ji}}{\partial R_i} = -V_{ij} \left(\frac{5}{2} + \frac{1}{2} \frac{1+3w_i}{1+3w_j} \right) \quad ; \tag{88}$$

which, in turn, produce:

$$R_u\frac{\partial\Omega_u}{\partial R_u}+R_v\frac{\partial\Omega_u}{\partial R_v}=-\Omega_u \quad ; \qquad u=i,j \quad ; \qquad v=j,i \quad ; \tag{89}$$

$$R_u\frac{\partial V_{uv}}{\partial R_u}+R_v\frac{\partial V_{uv}}{\partial R_v}=-V_{uv} \quad ; \qquad u=i,j \quad ; \qquad v=j,i \quad ; \tag{90}$$

owing to Eqs. (19), (89), and (90), the virial theorem for subsystems may be formulated as:

$$\begin{aligned}&2T_u-R_u\frac{\partial\Omega_u}{\partial R_u}-R_v\frac{\partial\Omega_u}{\partial R_v}-R_u\frac{\partial V_{uv}}{\partial R_u}-R_v\frac{\partial V_{uv}}{\partial R_v}=0 \quad ; \\ &u=i,j \quad ; \qquad v=j,i \quad ;\end{aligned} \tag{91}$$

and the sum of the two alternative expressions reads:

$$2T-R_i\frac{\partial\Omega}{\partial R_i}-R_j\frac{\partial\Omega}{\partial R_j}=0 \quad ; \tag{92a}$$

$$T=T_i+T_j \quad ; \tag{92b}$$

$$\Omega=\Omega_i+V_{ij}+V_{ji}+\Omega_j=\Omega_i+W_{ij}+W_{ji}+\Omega_j \quad ; \tag{92c}$$

where T and Ω, according to Eq. (22), are the kinetic and the potential energy, respectively, of the whole system.

The explicit expression of the last two terms on the left-hand side of Eq. (92a), and the related sum, may be calculated using Eqs. (82)-(88). The result is:

$$R_i\frac{\partial\Omega}{\partial R_i}=-\Omega_i-\frac{1}{2}\left(1+\frac{1+3w_i}{1+3w_j}\right)V_{ij} \quad ; \tag{93a}$$

$$R_j\frac{\partial\Omega}{\partial R_j}=-\Omega_j-\left(\frac{5}{4}y^2-\frac{3}{4}\right)\left(1+\frac{1+3w_i}{1+3w_j}\right)V_{ij} \quad ; \tag{93b}$$

$$R_i\frac{\partial\Omega}{\partial R_i}+R_j\frac{\partial\Omega}{\partial R_j}=-\Omega_i-\Omega_j-\left(\frac{5}{4}y^2-\frac{1}{4}\right)\left(1+\frac{1+3w_i}{1+3w_j}\right)V_{ij} \quad ; \tag{93c}$$

in the special case where the two subsystems fill the same volume, $R_i=R_j=R$ or $y=1$, Eq. (18) holds and Eqs. (92) reduce to:

$$\lim_{R_i\to R}\left(R_i\frac{\partial\Omega}{\partial R_i}\right)=-\Omega_i-\frac{1}{2}V_{ij}-\frac{1}{2}V_{ji} \quad ; \tag{94a}$$

$$\lim_{R_j \to R} \left(R_j \frac{\partial \Omega}{\partial R_j} \right) = -\Omega_j - \frac{1}{2} V_{ji} - \frac{1}{2} V_{ij} \quad ; \tag{94b}$$

$$\lim_{R_i \to R} \left(R_i \frac{\partial \Omega}{\partial R_i} \right) + \lim_{R_j \to R} \left(R_j \frac{\partial \Omega}{\partial R_j} \right) = -\Omega_i - V_{ij} - V_{ji} - \Omega_i = -\Omega \quad ; \tag{94c}$$

in conclusion, the total potential energy of the whole system must be conceived as dependent on four independent variables, $\Omega = \Omega(M_i, M_j, R_i, R_j)$.

On the other hand, one-component systems are subjected to no tidal potential, then the potential energy coincides with the self potential energy, which depends on two independent variables, $\Omega = \Omega(M, R)$. Accordingly, Eq. (92a) translates into:

$$2T - R \frac{\partial \Omega}{\partial R} = 0 \quad ; \tag{95}$$

which cannot be splitted as a sum of different contributions, as done in e.g., ML05, unless it is conceived as a function of four independent variables instead of two, and Eq. (92a) is used instead of Eq. (95).

B. The function $f(x)$

Let us define the function:

$$f(x) = \frac{1+x}{1+m} \, \frac{1+(1+3w)x}{1+(1+3w)m} \quad ; \tag{96a}$$

$$0 \le x \le m \quad ; \qquad 0 \le f(x) \le 2 \quad ; \qquad -1 \le w < -\frac{1}{3} \quad ; \tag{96b}$$

where the values at the extrema of the domain are:

$$f(0) = \frac{1}{1 + (2+3w)m + (1+3w)m^2} \quad ; \qquad f(m) = 1 \quad ; \tag{97}$$

and keeping in mind that the equation:

$$(1+x)[1+(1+3w)x] = 0 \quad ; \tag{98}$$

admits the real solutions:

$$x_1 = -1 \quad ; \qquad x_2 = -\frac{1}{1+3w} \quad ; \tag{99}$$

the sign of the function, $f(x)$, is determined by the inequalities:

$$f(x) > 0 \quad ; \qquad 0 \le m < \frac{-1}{1+3w} \quad ; \qquad 0 \le x < \frac{-1}{1+3w} \quad ; \tag{100a}$$

$$f(x) < 0 \quad ; \qquad m > \frac{-1}{1+3w} \quad ; \qquad 0 \le x < \frac{-1}{1+3w} \quad ; \tag{100b}$$

$$f(x) > 0 \quad ; \qquad m > \frac{-1}{1+3w} \quad ; \qquad \frac{-1}{1+3w} < x \le m \quad ; \tag{100c}$$

and the function is null at $x = -1$ and $x = -1/(1+3w)$, respectively. In the special cases, $m = -1$ and $m = -1/(1+3w)$, the function diverges everywhere within the domain, except at $x = m$ where $f(m) = 1$.

The first and the second derivatives are:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{(2+3w) + 2(1+3w)x}{1 + (2+3w)m + (1+3w)m^2} \quad ; \tag{101}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{2(1+3w)}{1 + (2+3w)m + (1+3w)m^2} \quad ; \tag{102}$$

where the first derivative is null at the extremum point:

$$x^\dagger = -\frac{1}{2}\frac{2+3w}{1+3w} \quad ; \tag{103}$$

and the sign of the second derivative is defined by the inequalities:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} < 0 \quad ; \qquad 0 \le m < \frac{-1}{1+3w} \quad ; \tag{104a}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} > 0 \quad ; \qquad m > \frac{-1}{1+3w} \quad ; \tag{104b}$$

in the range of interest, $-1 \le w < -1/3$. Accordingly, the extremum point, $x^\dagger$, is a maximum and a minimum, respectively.

At the extremum point, the function attains the value:

$$f(x^\dagger) = -\frac{9}{4}\frac{w^2}{1+3w}\frac{1}{1 + (2+3w)m + (1+3w)m^2} \quad ; \tag{105}$$

on the other hand, the extremum point is attained at the upper limit of the domain, $x^\dagger = m$, provided the parameter, w, has the value:

$$w = -\frac{2}{3}\frac{m+1}{2m+1} \quad ; \tag{106}$$

in particular, $w = -2/3$ for $x^\dagger = m = 0$ and $w = -1/3$ for $x^\dagger = m \to +\infty$. Within the range, $-1 \le w < -2/3$, the extremum point lies outside the domain, $x^\dagger < 0$.

To gain further insight, a few special cases shall be studied with more detail, namely $w = -1, -2/3, -1/2, -1/3$, the last to be conceived as an interesting limiting situation. The general case, $-1 \le w < -1/3$, is expected to show similar properties as in the closest among the above mentioned special cases.

In the special case, $w = -1/3$, Eq. (96a) reduces to:

$$f(x) = \frac{1+x}{1+m} \quad ; \tag{107}$$

which is positive within the domain, $0 \le x \le m$, according to Eq. (100a). The extremum point is a maximum and occurs at infinite, according to Eqs. (104a) and (105). Then it belongs to the domain only in the limit, $m \to +\infty$. The function, $f(x)$, for different values of the parameter, m, is represented in Fig. 7, top left.

In the special case, $w = -1/2$, Eq. (96a) reduces to:

$$f(x) = \frac{2+x-x^2}{2+m-m^2} \quad ; \tag{108}$$

which, according to Eqs. (100), is positive within the domain, $0 \le x \le m$, for $m < 2$, and within the domain, $2 < x \le m$, for $m > 2$; on the other hand, it is negative within the domain, $0 \le x < 2$, for $m > 2$. The (allowed) zero of the function occurs at $x = 2$. In the special case, $m = 2$, the function diverges everywhere within the domain, except at $x = 2$, where $f(2) = 1$.

Owing to Eq. (103), the extremum point occurs at $x^\dagger = 1/2$, which is a maximum for $0 \le m < 2$ and a minimum for $m > 2$, according to Eqs. (104). The value of the function at the extremum point is:

$$f(x^\dagger) = \frac{9}{4}\frac{1}{2+m-m^2} \quad ; \qquad x^\dagger = \frac{1}{2} \quad ; \tag{109}$$

which belongs to the domain only if $m \ge 1/2$, $m \ne 2$. The function, $f(x)$, for different values of the parameter, m, is represented in Fig. 7, top right.

In the special case, $w = -2/3$, Eq. (96a) reduces to:

$$f(x) = \frac{1-x^2}{1-m^2} \quad ; \tag{110}$$

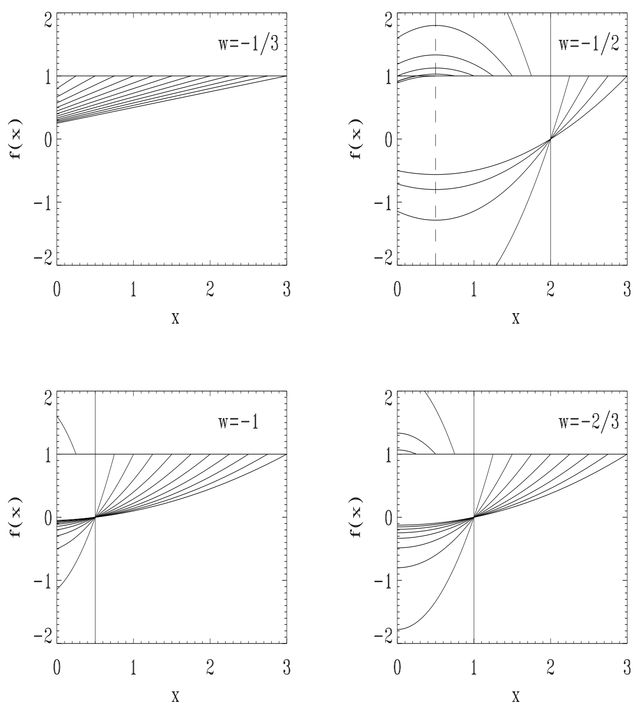

Figure 7: The function, $f(x)$, for different values of the parameter, m, within the domain, $0 \leq x \leq m$, with regard to special choices of the parameter, w, (from top left in the clockwise sense): $-1/3$, $-1/2$, $-2/3$, -1. The value of m related to each curve, starting from the left, is $m = i/4$, $0 \leq i \leq 12$. The horisontal axis corresponds to $m \rightarrow +\infty$. The locus of ending points is the horisontal line, $f(x) = 1$. In the special case, $m = 0$, the curve reduces to a single point, $(0, 1)$. The locus of extremum points is marked by a dashed vertical line for $w = -1/2$, and coincides with the vertical axis for $w = -2/3$.

which, according to Eq. (100), is positive within the domain, $0 \le x \le m$, for $m < 1$, and within the domain, $1 < x \le m$, for $m > 1$; on the other hand, it is negative within the domain, $0 \le x < 1$, for $m > 1$. The (allowed) zero of the function occurs at $x = 1$. In the special case, $m = 1$, the function diverges everywhere within the domain, except at $x = 1$, where $f(1) = 1$.

Owing to Eq. (103), the extremum point occurs at $x^\dagger = 0$, which is a maximum for $0 \le m < 1$ and a minimum for $m > 1$, according to Eqs. (104). The value of the function at the extremum point is:

$$f(x^\dagger) = \frac{1}{1-m^2} \quad ; \qquad x^\dagger = 0 \quad ; \tag{111}$$

which, in any case, belongs to the domain. The function, $f(x)$, for different values of the parameter, m, is represented in Fig. 7, bottom right.

In the special case, $w = -1$, Eq. (96a) reduces to:

$$f(x) = \frac{1 - x - 2x^2}{1 - m - 2m^2} \quad ; \tag{112}$$

which, according to Eqs. (100), is positive within the domain, $0 \le x \le m$, for $m < 1/2$, and within the domain, $1/2 < x \le m$, for $m > 1/2$; on the other hand, it is negative within the domain, $0 \le x < 1/2$, for $m > 1/2$. The (allowed) zero of the function occurs at $x = 1/2$. In the special case, $m = 1/2$, the function diverges everywhere within the domain, except at $x = 1/2$, where $f(1/2) = 1$.

Owing to Eq. (103), the extremum point occurs at $x^\dagger = -1/4$, which is outside the domain. The function, $f(x)$, for different values of the parameter, m, is represented in Fig. 7, bottom left.

C. The effect of partially clustered quintessence on the virialized configuration

With regard to a virialized configuration in the special case of fully clustered quintessence, let $\Delta\Omega$ be the energy change, $E - E_{\mathrm{FC}}$, which defines the virialized configuration in the case of partially clustered quintessence, related to $\Delta\mu = \zeta \eta_{\mathrm{FC}}^{-3w}$, according to Eqs. (54)-(57), (65), and (69). The combination of Eqs. (56), (65), and (69) yields:

$$\Delta\Omega = -\frac{3}{5}\frac{GM_m^2}{R_{\max}}\frac{1}{\eta_{\mathrm{FC}}}\left[(1+3w)\zeta(2-\zeta)x^2 + (2+3w)\zeta x\right] \quad ; \tag{113a}$$

$$x = m\eta_{\rm FC}^{-3w} \quad ; \qquad 0 \le x \le m \quad ; \tag{113b}$$

and the substitution of Eq. (113b) into (70) produces:

$$\phi(x; w, \zeta) = (1+3w)\zeta(2-\zeta)x^2 + (2+3w)\zeta x \quad ; \tag{114}$$

which makes Eq. (113a) be cast into the form:

$$\Delta\Omega = -\frac{3}{5}\frac{GM_m^2}{R_{\rm max}}\frac{1}{\eta_{\rm FC}}\phi \quad ; \tag{115}$$

accordingly, the sign of the energy change, $\Delta\Omega$, is opposite to the sign of the function, $\phi(x)$. Keeping in mind that positive and negative $\Delta\Omega$ imply expansion and contraction, respectively, with regard to the virialized configuration in the special case of fully clustered quintessence, the effect of partial clustering may be deduced from the sign of $\phi(x)$. According to Eq. (114), the solutions of $\phi(x) = 0$ are:

$$x_1 = 0 \quad ; \qquad x_2 = -\frac{1}{2-\zeta}\frac{2+3w}{1+3w} \quad ; \tag{116}$$

and the sign of $\phi(x)$ is defined as:

$$\phi(x) \ge 0 \quad ; \qquad \min(x_1, x_2) \le x \le \max(x_1, x_2) \quad ; \tag{117a}$$

$$\phi(x) \le 0 \quad ; \qquad x \le \min(x_1, x_2) \quad ; \qquad x \ge \max(x_1, x_2) \quad ; \tag{117b}$$

where $0 \le x \le m$ in the case under discussion, conform to Eq. (96).

The first derivative:

$$\frac{{\rm d}\phi}{{\rm d}x} = 2\zeta(2-\zeta)(1+3w)x + \zeta(2+3w) \quad ; \tag{118}$$

is null at the abscissa:

$$x^\dagger = -\frac{1}{2}\frac{1}{2-\zeta}\frac{2+3w}{1+3w} \quad ; \tag{119}$$

where, owing to Eq. (116), $|x^\dagger - x_1| = |x^\dagger - x_2|$, or $x^\dagger = x_2/2$.

The second derivative:

$$\frac{{\rm d}^2\phi}{{\rm d}x^2} = 2\zeta(2-\zeta)(1+3w) \quad ; \tag{120}$$

is negative within the range of interest, $-1 \leq w < -1/3$, $0 \leq \zeta \leq 1$, which implies that the extremum point is a maximum. The substitution of Eq. (119) into (114) allows the calculation of the function at the maximum. The result is:

$$\phi(x^\dagger) = -\frac{1}{4}\frac{\zeta}{2-\zeta}\frac{(2+3w)^2}{1+3w} \quad ; \tag{121}$$

which is non negative, $\phi(x^\dagger) \geq 0$, in the case under discussion.

Further inspection of Eq. (119) shows that:

$$x^\dagger > 0 \quad ; \qquad -\frac{2}{3} < w < -\frac{1}{3} \quad ; \tag{122a}$$

$$x^\dagger < 0 \quad ; \qquad -1 < w < -\frac{2}{3} \quad ; \tag{122b}$$

$$x^\dagger = 0 \quad ; \qquad w = -\frac{2}{3} \quad ; \tag{122c}$$

$$x^\dagger \to +\infty \quad ; \qquad w \to \left(-\frac{1}{3}\right)^- \quad ; \tag{122d}$$

independent of the value of ζ within the assigned range.

The particularization of Eqs. (114), (116), (119), and (121), to a few special values of ζ, yields:

$$\phi(x) = \zeta x \quad ; \tag{123a}$$

$$\phi(x^\dagger) \to +\infty \quad ; \qquad x^\dagger = \frac{1}{2}x_2 \to +\infty \quad ; \tag{123b}$$

in the special case, $w = -1/3$;

$$\phi(x) = -\frac{1}{2}\zeta\left[(2-\zeta)x^2 - x\right] \quad ; \tag{124a}$$

$$\phi(x^\dagger) = \frac{1}{8}\frac{\zeta}{2-\zeta} \quad ; \qquad x^\dagger = \frac{1}{2}x_2 = \frac{1}{2}\frac{1}{2-\zeta} \quad ; \tag{124b}$$

in the special case, $w = -1/2$;

$$\phi(x) = -\zeta(2-\zeta)x^2 \quad ; \tag{125a}$$

$$\phi(x^\dagger) = 0 \quad ; \qquad x^\dagger = \frac{1}{2}x_2 = 0 \quad ; \tag{125b}$$

in the special case, $w = -2/3$;

$$\phi(x) = -\zeta \left[2(2-\zeta)x^2 + x \right] \quad ; \tag{126a}$$

$$\phi(x^\dagger) = \frac{1}{8}\frac{\zeta}{2-\zeta} \quad ; \qquad x^\dagger = \frac{1}{2}x_2 = -\frac{1}{4}\frac{1}{2-\zeta} \quad ; \tag{126b}$$

in the special case, $w = -1$.

The function, $\phi(x)$, expressed by Eqs. (123a)-(126a) and related to different values of ζ, are represented in Fig. 8 with regard to the special cases considered above.

Keeping in mind Eq. (115), the transition from a virialized configuration related to fully clustered quintessence to its counterpart related to partially clustered quintessence, implies expansion ($\phi < 0$) for values of the quintessence equation of state parameter, w, in the range, $-1 \leq w < -2/3$. On the other hand, both expansion and contraction ($\phi > 0$) may occur for $-2/3 \leq w < -1/3$, according if the variable, x, is sufficiently distant from 0 and/or the parameter, w, is sufficiently close to $-2/3$, and vice versa.

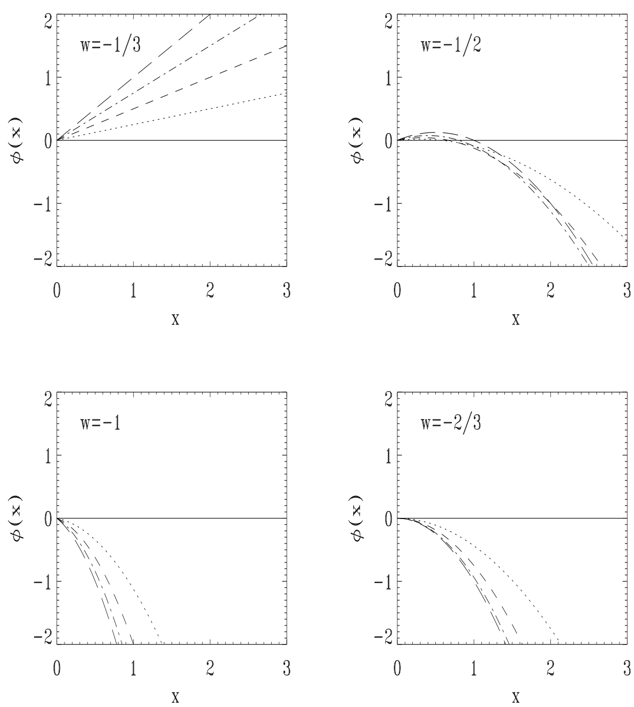

Figure 8: The function, $\phi(x)$, for different values of the parameter, ζ, with regard to special choices of the parameter, w, (from top left in the clockwise sense): $-1/3$, $-1/2$, $-2/3$, -1. Dotted curves correspond to $\zeta = 1/4$, dashed curves to $\zeta = 1/2$, dot-dashed curves to $\zeta = 3/4$, and long-dashed curves to $\zeta = 1$, which is the limit of fully de-clustered quintessence. In the limit of fully clustered quintessence, $\zeta = 0$, all curves coincide with the horisontal axis.